J. Fluid Mech. (1969), *vol.* 37, *part* 1, *pp.* 31–50
Printed in Great Britain

Non-linear wave propagation in a relaxing gas

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(Received 3 September 1968)

An outline of the classical far- and near-field solutions for small-amplitude one-dimensional unsteady flows in a general inviscid relaxing gas is given. The structure of the complete flow field, including a non-linear near-frozen (high frequency) region at the front, is obtained by matching techniques when the relaxation time is 'large'.

If the energy in the relaxing mode is small compared with the total internal energy, the solution in the far field is, in general, more complex than that predicted by classical theory. In this case the rate process is not necessarily able to diffuse all convective steepening. An equation valid in this limit is derived and discussed. In particular, a sufficient condition for the flow to be shock-free is established. For an impulsively withdrawn piston it is shown that the solution is single-valued both within and downstream of the fan. Some useful similarity rules are pointed out.

The corresponding formulation for two-dimensional steady flows is also noted in the small energy limit.

1. Introduction

It has long been suggested that the far-field behaviour for small-amplitude motions in a relaxing gas is governed by Burger's equation (Lighthill 1956; Lick 1965) in which the principal signal is centred on the equilibrium or low-frequency characteristics. In §§ 2 and 3 of this paper a brief review of the classical results for small-amplitude non-equilibrium flow is given for a gas with general thermodynamic properties. It is assumed that the rate of change of energy in the lagging mode depends only on the local values of the pressure, density and internal energy. The size and position of the region in which Burger's equation holds is deduced in terms of an amplitude parameter.

If, however, the relaxation time is much larger than the time scale associated with the piston signal, there is also a region in the neighbourhood of the front where non-linear convection associated with the high-frequency characteristics is important (Varley & Rogers 1967). On the front itself it is possible to obtain an exact solution, over a certain time interval, for the derivatives normal to the front (Whitham 1959; Rarity 1967).

The relation of the classical results to this high-frequency limit is outlined in § 4. Far behind the front this solution can be matched to the solution of the

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classical linearized equation discussed in § 2. For still larger times, away from the front, the linearized result can be shown to match with Burger's equation which describes the principal asymptotic wave structure. The non-linear near-frozen solution is also invalid at very large times in the neighbourhood of the front, where the solution is governed locally by the telegraph equation.

For the corresponding one-dimensional unsteady flow of a perfect gas it is well known that the usual linearized result can be made uniformly valid by formally replacing the linearized characteristics by the exact ones (see e.g. Lighthill 1955). Clarke (1965) has attempted to apply this technique to the non-equilibrium situation but noted that the approach met with certain difficulties. Since at large times far from the piston face, and in the equilibrium limit, any non-linear convection is associated with the low-frequency characteristics (see § 3), Clarke's conclusions are not surprising: replacing the linearized characteristics by the complete (high-frequency) characteristics only corrects for non-linear convection associated with these latter wavelets.

An important feature of these theories is that any discontinuities predicted by the front solution are ultimately exponentially weak, and consequently any asymptotic steady state can be described as fully dispersed, as opposed to the more commonly occurring partly dispersed situation in which the relaxation region is preceded by a Rankine–Hugoniot shock (Lighthill 1956). This conclusion follows directly from the assumption inherent in the theories that the energy in the relaxing mode is comparable with the total internal energy, and hence that the difference in the high- and low-frequency sound speeds is not 'small'.

If this latter assumption is discarded and both the lagging energy and the amplitude are assumed to be small, in some appropriate sense, it can be shown (see § 5) that the governing equations can be reduced to the form

$$(\mathscr{V}w)_\psi + \mathscr{V}w + w_\psi = 0, \tag{1.1}$$

where $\mathscr{V}$ is the non-linear operator

$$\frac{\partial}{\partial Y} + w\frac{\partial}{\partial \psi},$$

Y and ψ are appropriate independent co-ordinates, and w is proportional to the fluid speed.

It is significant that both the high- and low-frequency convective terms are non-linear. In fact, it is easily shown that the high-frequency behaviour of this equation includes the near-frozen Varley–Rogers result and the low-frequency behaviour is governed, to a certain order, by Burger's equation. Moreover, it is shown in § 6 that partly dispersed steady-state solutions of (1.1) do exist.

Apart from these steady-state solutions, analytical solutions of (1.1) are not readily found; but the equation possesses several interesting features which make it worthy of further study. Certainly (1.1) does not appear to have been derived elsewhere in the literature.†

Obviously this hyperbolic equation can be solved by numerical methods and it is of interest to note that in certain cases the boundary conditions depend on

† However, Spence & Ockendon (1968) have recently obtained a similar equation independently.

only a single parameter ((1.1) is already written in a form free of parameters). Since the analysis is for a general relaxing gas these similarity rules are particularly appealing. Moreover, for a centred expansion wave, within the fan, the problem can be expressed so that it is independent of all parameters. This similarity form is discussed in § 5 and § 7, and a numerical solution is given in § 7.

The situation in which the complete flow field is shock-free, in the unsteady case, is of some interest (Rarity 1967). A simple criterion which ensures that the solution of (1.1) is unique is given in § 7 for non-centred waves.

Some exact numerical solutions of the full equations (Mohammad 1967) for a centred expansion wave, produced by impulsively withdrawing a piston in a vibrationally relaxing gas, have suggested the existence of shocks on or immediately downstream of the tail of the fan. It is shown in § 7 for the non-linear small energy limit that the solution is single-valued both within and downstream of the fan.

For simplicity, only one-dimensional unsteady flows are considered in detail in the main body of this paper but the results for two-dimensional steady supersonic flow are essentially the same. A brief outline of the appropriate non-linear equation in the small energy limit is given in § 8, and the corresponding similarity forms are noted.

2. The characteristic relations and classical linearized theory

The governing equations for the one-dimensional unsteady flow of a relaxing gas can be written in characteristic form, with respect to a co-ordinate system fixed in space, as (Broer 1958)

$$\frac{D_{\pm}p}{Dt} \pm \rho a \frac{D_{\pm}u}{Dt} = -c\rho \frac{D\sigma}{Dt}, \tag{2.1}$$

together with the energy equation

$$\frac{Dp}{Dt} - a^2 \frac{D\rho}{Dt} + c\rho \frac{D\sigma}{Dt} = 0. \tag{2.2}$$

The frozen sound speed a is defined by

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_{S,\sigma} = -\frac{\rho h_\rho}{\rho h_p - 1} \tag{2.3}$$

and

$$c = \frac{h_\sigma}{\rho h_p - 1}. \tag{2.4}$$

Here the enthalpy $h = h(p, \rho, \sigma)$, where σ is the energy in the lagging mode. In (2.1) and (2.2) D/Dt is the usual convective operator and

$$\frac{D_{\pm}}{Dt} = \frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}, \tag{2.5}$$

where t is the time and x is a suitable space co-ordinate. p, ρ, and u are the pressure, density and velocity respectively. All independent and dependent variables are suitably non-dimensionalized (see appendix).

The right-hand side of (2.1) is known as a local function of p, ρ and σ from the rate equation

$$D\sigma/Dt = \Lambda F(p,\rho,\sigma), \tag{2.6}$$

where Λ is the rate parameter. In thermodynamic equilibrium ($\sigma = \bar{\sigma}(p,\rho)$)

$$F(p,\rho,\bar{\sigma}) = 0. \tag{2.7}$$

It is assumed that the partial derivative F_σ exists and is non-zero, so that for small deviations from any initial equilibrium state F is linearly dependent on the departure from equilibrium.

The general problem considered here is defined by the class of piston motions for which the amplitude of the disturbance is, in some suitable sense, 'small', though the initial acceleration is finite. (This latter restriction can be removed and the solution for an impulsively started piston obtained.) The piston path is defined by

$$x = \delta f(t) \quad (t \geqslant 0), \tag{2.8a}$$

where $\delta \ll 1$.† For the finite acceleration case

$$|f| \sim \tfrac{1}{2}t^2 \quad (t \to 0). \tag{2.8b}$$

It is assumed throughout this paper that the initial conditions correspond to an equilibrium state.

The linearized equation associated with this and related problems has been discussed many times in the literature (see e.g. Clarke & McChesney 1964). This equation describes the solution for x, $t = O(1)$ and is easily derived from the preceding equations by means of the expansion

$$\left.\begin{aligned} u(x,t;\,\delta) &= \delta u_1(x,t) + \ldots, \\ p(x,t;\,\delta) &= 1 + \delta p_1(x,t) + \ldots, \\ \sigma(x,t;\,\delta) &= \bar{\sigma}_0 + \delta\sigma_1(x,t) + \ldots, \end{aligned}\right\} \tag{2.9}$$

etc.

Substitution into (2.1), (2.2) and (2.6) leads to the following equation for the first-order perturbation u_1:

$$\frac{\partial}{\partial t}(u_{1_{tt}} - a_0^2 u_{1_{xx}}) + \lambda(u_{1_{tt}} - \bar{a}_0^2 u_{1_{xx}}) = 0, \tag{2.10}$$

where

$$\bar{a}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{S,\sigma=\bar{\sigma}} = \frac{-\rho\bar{h}_\rho}{\rho\bar{h}_p - 1} \tag{2.11}$$

is the equilibrium sound speed and $\bar{h} = h(p,\rho,\bar{\sigma})$ is the equilibrium enthalpy. In addition

$$\bar{c} = h_\sigma/(\rho\bar{h}_p - 1) \tag{2.12}$$

and

$$\lambda = \Lambda(-F_\sigma c/\bar{c})_0 \tag{2.13}$$

is a modified rate parameter. The suffix 0 denotes evaluation with respect to the initial conditions. From (2.8) the appropriate linearized boundary condition is

$$u_1 = f'(t) \quad \text{on} \quad x = 0. \tag{2.14}$$

† The generalization to $x = \delta f(t;\,\delta)$ is straightforward.

It is convenient to replace (x, t) by the independent variables (y, ξ), where

$$y = 2x/a_0, \quad \xi = t - x/a_0; \tag{2.15}$$

ξ is the linearized (frozen) characteristic associated with (2.10). In terms of these variables (2.10) becomes identical in form with the equation discussed by Clarke (1965), and many of Clarke's results are directly transferable. A formal solution can be obtained by Laplace transforms but the most important result, with a view to later application, is the behaviour of the solution for large time *far* from the piston face. This asymptotic behaviour, $y \to \infty$, is (Clarke 1965; Whitham 1959)

$$u_1 \sim \int_0^{\bar{\xi}} \frac{f'(s)}{\sqrt{(\pi \Gamma y\{\alpha - (1/\alpha)\})}} \exp\left\{-\frac{(\bar{\xi} - s)^2}{\Gamma y\{\alpha - (1/\alpha)\}}\right\} ds, \tag{2.16}$$

where
$$\alpha = a_0/\bar{a}_0, \quad \Gamma = \alpha^2/\lambda \tag{2.17}$$

and
$$\bar{\xi} = t - x/\bar{a}_0 \tag{2.18}$$

is the linearized (equilibrium) characteristic associated with the low-frequency (large time) behaviour of (2.10).

3. The low-frequency solution

At large time, far from the piston face (i.e. $y \gg 1$), non-linear convection may be important. It is usually assumed that the behaviour in this 'front' region is described by Burger's equation (Lighthill 1956; Jones 1964). In the present section the conditions under which this equation does govern the asymptotic state are briefly discussed.

In deriving Burger's equation it is convenient to replace σ by the departure from equilibrium

$$\epsilon = \sigma - \bar{\sigma} \tag{3.1}$$

and to regard the enthalphy etc. as functions of p, ρ and ϵ.

The principal signal in this outer region is centred on the equilibrium characteristic (see (2.16)). If $u = O(\Delta(\delta))$ for $y \gg 1$, the only non-trivial scaling of the independent variables is

$$X = \Delta^2(\delta)\, y, \quad \bar{Z} = \Delta(\delta)\, \bar{\xi} \tag{3.2}$$

and the dependent variables have expansions

$$\left.\begin{aligned} u &= \Delta(\delta)\, U_1(X, \bar{Z}) + \ldots, \\ p &= 1 + \Delta(\delta)\, P_1(X, \bar{Z}) + \ldots, \\ \epsilon &= \Delta^2(\delta)\, E_2(X, \bar{Z}), \end{aligned}\right\} \tag{3.3}$$

etc. Substitution in the governing equations yields after some simplification

$$U_X - U U_{\bar{Z}} = \frac{\Gamma}{4}\left(\alpha - \frac{1}{\alpha}\right) U_{\bar{Z}\bar{Z}}, \tag{3.4}$$

where
$$U = \bar{b}/a_0 U_1, \tag{3.5}$$

and
$$\bar{b} = \tfrac{1}{2}\alpha^2 \left\{\frac{1}{\bar{a}}\left(\frac{\partial}{\partial \rho}\rho \bar{a}\right)_{S, \epsilon}\right\}_0. \tag{3.6}$$

In this derivation of Burger's equation (3.4) it has been assumed that $U_1 = O(1)$. However, as discussed below, the stretching (3.3) is not permissible for all piston paths.

$\Delta(\delta)$, the magnitude of the velocity in the far field, is defined implicitly by the outer expansion of the inner linearized solution. This expansion (2.16) can be written, neglecting terms which are exponentially small, in terms of outer variables as

$$U_1 \sim \frac{\delta}{\Delta(\delta)}\int_0^\infty \frac{g(S;\,\delta)}{\sqrt{(\pi\Gamma X\{\alpha-(1/\alpha)\})}}\exp\left\{-\frac{(\bar{Z}-S)^2}{\Gamma X\{\alpha-(1/\alpha)\}}\right\}dS. \tag{3.7}$$

Here δg is the piston speed defined in outer variables, i.e.

$$f'(\bar{\xi}) = g(\bar{Z};\,\delta). \tag{3.8}$$

f' is, by definition, at most $O(1)$ for all time. For decaying motions it is apparent that $g = o(1)$ as $\delta \to 0$, though $g(\bar{Z};\,\delta)$ cannot always be replaced by its asymptotic expansion as $\delta \to 0$, since the integrand is then not necessarily well behaved at the origin (which implies that significant contributions to the value of the integral may come from this region). In fact if

$$f'(t) \sim t^{-n} \quad (n \geqslant 0) \tag{3.9}$$

as $t \to \infty$, the integral in (3.7) is $O(\Delta^n)$ for $n < 1$. Hence $\Delta = O(\delta^{1/(1-n)})$.

If $n > 1$ the corresponding integral (subject to (2.8*b*)) is $O(\Delta)$ and $U = O(\delta)$, which contradicts the original assertion. In such a case the non-linear terms are not important in this region and the linear theory, defined by the diffusion equation, remains valid in the far field. A similar conclusion apparently holds in general for $y \gg 1$ in the low-frequency front region, when the piston displacement is bounded.

In general, the condition (3.7) is equivalent to the solution of the diffusion equation, with a diffusivity $\frac{1}{4}\Gamma\{\alpha-(1/\alpha)\}$, for a source distribution of strength per unit length

$$(\delta/\Delta)\,g(Z;\,\delta) = G(\bar{Z};\,\delta) \tag{3.10}$$

along $Y = 0$, for $0 < \bar{Z} < \infty$ (see Whitham 1959). (For the simple case (3.10), $n < 1$, $G = \bar{Z}^{-n}$.) Since the inner behaviour of (3.4) is obviously given by this diffusion equation the solution of (3.4) will match with (3.7) if it satisfies the boundary condition (3.10) on $Y = 0$. The appropriate general solution of Burger's equation can be found in Lighthill (1956).

A significant feature of this solution in the far field is its continuous structure. Any convective steepening is always diffused by the dissipative nature of the relaxation. It follows that, in this region, the behaviour can always be described as fully dispersed.

4. The high-frequency limit

The low-frequency solution discussed in § 3 is valid for large time when $\Lambda = O(1)$ or greater. If, however, $\Lambda \to 0$ non-linear convection is now associated with the high-frequency characteristics

$$dx/dt = u \pm a.$$

For $\Lambda \equiv 0$, the large time solution is the perfect gas far-field theory (see e.g. Lighthill 1955). For $\Lambda = O(\delta)$ there is a non-trivial solution with $x = O(\delta^{-1})$, which is closely related to a problem discussed recently by Varley & Rogers (1967), who considered wave propagation in a visco-elastic material. Varley & Rogers used an exact characteristic co-ordinate as one independent variable. Here the problem is considered directly in (ξ, η) space, where

$$\eta = 2\delta x/a_0 = \delta y \tag{4.1}$$

and ξ was defined in (2.15).

In this section a further non-uniformity in the high-frequency solution at the front is noted and discussed. It is also apparent that behind the front the low-frequency terms must eventually become important. Some comments are made regarding the structure of the complete flow field, and the domains of validity of the various solutions are outlined.

Near the front ($\xi = O(1)$) the variables are expanded in the form

$$\left.\begin{aligned} u(x,t;\delta) &= \delta q_1(\xi,\eta)+\ldots, \\ p(x,t;\delta) &= 1+\delta\pi_1(\xi,\eta)+\ldots, \\ \sigma(x,t;\delta) &= \overline{\sigma}_0+\delta^2\Sigma_2(\xi,\eta)+\ldots,\end{aligned}\right\} \tag{4.2}$$

and substitution in (2.1), (2.2) and (2.6) yields

$$q_\eta - qq_\xi + kq = 0, \tag{4.3}$$

where

$$\left.\begin{aligned} q &= bq_1/a_0, \\ b &= \frac{1}{2}\left\{\frac{1}{\alpha}\left(\frac{\partial\rho a}{\partial\rho}\right)_{S,\sigma}\right\}_0\end{aligned}\right\} \tag{4.4}$$

and

$$k = (\alpha^2-1)/4\delta\Gamma. \tag{4.5}$$

Note that, since $\Lambda = O(\delta)$, $k = O(1)$.

The inner linearized solution when $\Lambda = O(\delta)$ is, to first order, the usual perfect gas (frozen) result

$$q_1 = f'(\xi) \tag{4.6}$$

and the solution of (4.3) which matches with (4.6) is

$$q_1 = f'(\phi)\,e^{-k\eta}. \tag{4.7}$$

ϕ is a parameter which is identified with the characteristic surface defined by

$$\left(\frac{\partial\xi}{\partial\eta}\right)_\phi = -q. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\xi = \phi-(b/a_0k)f'(\phi)\,[1-e^{-k\eta}] \tag{4.9}$$

if $\phi = \xi$ on $\eta = 0$.

In contrast to the low-frequency non-linear solution discussed earlier, the high-frequency solution does permit discontinuities or shocks. The solution defined by (4.7) and (4.9) is not single-valued in the physical (ξ,η) space at points where

$$\xi_\phi = 0 \tag{4.10}$$

or
$$\left.\begin{aligned} \eta = \eta_c &= -\frac{1}{k}\log\left(1-\frac{ka_0}{bf''(\phi)}\right), \\ \xi = \xi_c &= \phi - \frac{f'(\phi)}{f''(\phi)} \end{aligned}\right\} \tag{4.11}$$
(see also Varley & Rogers 1967).

For piston paths such that
$$\frac{ka_0}{b[f''(\phi)]_{\max}} > 1 \tag{4.12}$$
the solution is single-valued and, to this order of approximation, remains shock free. The results corresponding to (4.11) and (4.12) on the front $\phi = 0$, where (4.11) is exact, were pointed out by Rarity (1967). It is apparent that (4.12) will always hold, for a given piston path, if k (or Λ) is sufficiently large. In such cases any convective steepening of the wave-form is again completely balanced by the diffusive effects of the rate process. This result is in accord with the low-frequency solution (§ 3) for $\Lambda = O(1)$.

When the shock forms at the front and propagates into an undisturbed region its path is easily determined. The Rankine–Hugoniot equations, together with
$$[\sigma] = 0,$$
lead to the usual relations between the pressure, density and velocity perturbations. In addition the shock speed U_s is related to q by
$$(U_s/a_0) - 1 = \delta q + \dots. \tag{4.13}$$
Using this result, with (4.7) and (4.9), it can be shown that the shock path is defined parametrically by
$$\left.\begin{aligned} \eta &= -\frac{1}{k}\log\left(1-\frac{2a_0 k}{b}f(\phi)/f'^2(\phi)\right), \\ \xi &= \phi - 2f(\phi)/f'(\phi). \end{aligned}\right\} \tag{4.14}$$

These results, in particular, imply that the asymptotic position of the shock front is ahead of $x = a_0 t$ by the fixed amount
$$(b/k)f'(\phi)_0 - a_0\phi_e, \tag{4.15}$$
where ϕ_e is defined by
$$f(\phi_e)/f'^2(\phi_e) = b/2ka_0. \tag{4.16}$$
Note, however, that the shock is exponentially weak in this limit. In figure 1 a typical variation of the shock strength is sketched for progressive piston motions which are governed initially by (2.8*b*) but which asymptotically approach a constant speed.

Even apart from any shock formation the near-frozen expansion (4.2) is not uniformly valid throughout the front region. In particular, higher-order terms suggest that the non-uniformity, for $\phi = O(1)$, occurs when $\eta = O(\delta^{-1})$ where the velocity is exponentially small. The formal expansion
$$u = e^{-k\eta}Q_1(\xi,\eta;\delta) + e^{-2k\eta}Q_2(\xi,\eta;\delta) + \dots \tag{4.17}$$

shows that neglecting exponentially smaller terms u satisfies the usual linearized equation of § 2. For the near-frozen limit considered here

$$Q_1(\xi, \eta; \delta) = \delta Q_{11}(\xi, \eta_1) + \dots, \tag{4.18}$$

where
$$\eta_1 = \delta\eta \tag{4.19}$$

and Q_{11} satisfies
$$Q_{11_{\xi\eta}} - KQ_{11} = 0, \tag{4.20}$$

with
$$K = (k/\delta\Gamma)\{1 + \tfrac{1}{4}(\alpha^2 - 1)\}. \tag{4.21}$$

In deriving (4.20) the appropriate front conditions have been used and, for simplicity, it is assumed in this particular discussion that $1 > b/a_0 k$.

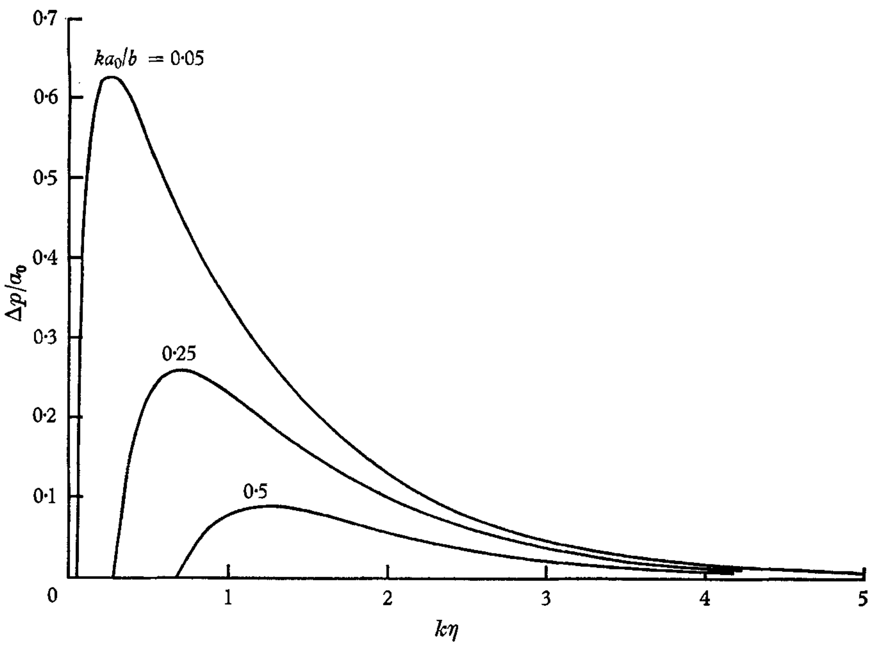

FIGURE 1. Variation in shock strength in the high-frequency limit. The piston path is defined by $f = \frac{1}{2}t^2/(1+t)$.

Initial conditions for the solution of the telegraph equation (4.20) are provided by matching with the non-linear front solution (4.7) and (4.9). It follows that

$$Q_{11} \sim f'(\phi(\xi)) \quad \text{as} \quad \eta_1 \to 0, \tag{4.22}$$

where $\phi(\xi)$ is defined by

$$\xi = \phi - (b/a_0 k) f'(\phi). \tag{4.23}$$

The appropriate solution of (4.20) is

$$Q_{11} = \int_0^\xi I_0(2K^{\frac{1}{2}}\eta_1^{\frac{1}{2}}S^{\frac{1}{2}}) f'\{\phi(\xi) - \phi(S)\}\, dS \tag{4.24}$$

with the usual notation for Bessel functions.

In addition to any non-uniformity for η large the front solution is also not valid as $\phi \to \infty$, where the low-frequency terms are again important. When $\eta = O(1)$ and $\phi = O(\delta^{-1})$, for $\Lambda = O(\delta)$, it is easily shown that the solution is governed by the linearized equation of § 2. Moreover, (4.9) shows that $\xi \sim \phi$, for ϕ large, provided that $f'(\phi)$ is bounded, and thus from (4.7)

$$q \sim f'(\xi)\, e^{-k\eta} \tag{4.25}$$

in this limit. Equation (4.25) can be shown to be compatible with the behaviour of the linearized equation at the front (Clarke 1965). In fact it follows that, if the linearized characteristic variable ξ is replaced by the 'exact' variable ϕ in the linearized solution, the solution is rendered uniformly valid for $\eta = O(1)$. This result, which can be regarded as an application of the PLK technique, has some bearing on a conjecture made by Clarke (1965) that replacing the linearized characteristics by the exact ones, when $\Lambda = O(1)$, might give some information about the far-field solution, as it does in the classical problem for a perfect gas.

x	ξ	Governing equation
$O(1)$	$O(1)$	Linearized perfect gas flow
$O(\delta^{-1})$	$O(1)$	Varley–Rogers
$O(\delta^{-2})$	$O(1)$	Telegraph
$O(\delta^{-1})$	$O(\delta^{-1})$	Classical linear (2.10)
$O(\delta^{-1}\Delta_1^{-2})$	$O(\delta^{-1}\Delta_1^{-1})$	Burger

TABLE 1. Solution régimes in the near-frozen limit $\Lambda = O(\delta)$

Since, as observed in § 3, the principal signal for large time is centred around the equilibrium characteristics, it is not surprising that Clarke's approach may break down in the far field when $\Lambda = O(1)$, though, as outlined above, the approach does apply for $\eta = O(1)$ when $\Lambda = O(\delta)$. Moreover, similarly to the discussion in § 3, if the piston displacement is unbounded this linearized solution is not uniformly valid even for $\Lambda = O(\delta)$ and non-linear convection is again important when both ϕ and η are large. More precisely, if $u = O(\Delta_1(\delta))$ in this region appropriate independent variables are

$$\left.\begin{aligned} \Delta_1^2\eta &= \delta\Delta_1^2(2x/a_0), \\ \Delta_1\delta\bar{\xi} &= \delta\Delta_1(t-x/\bar{a}_0), \end{aligned}\right\} \tag{4.26}$$

and the flow is governed, as expected, by Burger's equation. $\Delta_1(\delta)$ is again (see § 3) defined implicitly by the outer expansion (2.16) of the linearized solution expressed in appropriate variables.

Since any shocks defined by the high-frequency solution are exponentially weak for $\eta \gg 1$ there is no contradiction between these latter results, which imply that the solution is shock free, and those outlined earlier, provided that exponentially small terms are neglected.

The conclusions of this section are summarized in table 1. If $\Lambda = O(1)$ the classical linearized solution is valid between the body surface and the front; the non-linear high-frequency solution then collapses onto the front.

5. The small energy far-field equation

One of the principal features of the analysis outlined in §§ 2–4 has been the result that either the solution is shock-free or that, if a shock is formed, its strength decays exponentially. Thus the wave-form, in some asymptotic sense, is always fully dispersed. However, it is well known that stable partly dispersed wave-forms do exist and it is pertinent to discuss this limitation in the preceding solutions.

In each of these solutions it was assumed that the energy σ in the inert mode was of the same magnitude as the total internal energy, but that the amplitude of the disturbance was small. Since the overall effect of the rate process is to diffuse the piston signal it is not surprising, when $\sigma = O(1)$ but $u = O(\delta)$, that any convective steepening can be balanced by the relaxation. Moreover, for $\sigma = O(1)$, the difference in sound speeds $a - \bar{a} = O(1)$ and it follows, e.g. in the steady-state case, that a partly dispersed wave-form can only exist if the piston speed is not 'small'. Obviously a discussion of the case $u = O(1)$, $\sigma = O(1)$ requires a solution of the full equations (for steady one-dimensional flow see e.g. Johannesen 1961) but the case $u = O(\delta)$, $\sigma = O(\delta)$† does have some interesting features. Since the internal energy in the relaxing mode is now small it can no longer be expected that the rate process will always be able to diffuse all steepening of the wave-form.

In the small energy limit the inner linearized solution for $x, t = O(1)$ is given, as in § 4, by the perfect gas result

$$u = \delta f'(\xi). \tag{5.1}$$

Appropriate independent variables in the far field are again ξ and η (see § 4). For the energy distribution

$$\sigma = \delta(\bar{e}_{\nu_0} + \delta e_{\nu_1} + \ldots), \tag{5.2}$$

where

$$\bar{e}_{\nu_0} = \bar{\sigma}_0/\delta \tag{5.3}$$

and the e_{ν_i} are $O(1)$. In addition

$$u = \delta v_1(\xi, \eta) + \ldots \tag{5.4}$$

and other dependent variables have similar expansions to those outlined in § 4.

Substitution of these expansions into (2.1), (2.2) and (2.6) yields

$$p_1 = a_0^2 \rho_1 = a_0 v_1 \tag{5.5}$$

and

$$v_\eta - v v_\xi = -e_{\nu_\xi}, \tag{5.6}$$

$$e_{\nu_\xi} = kv - \lambda e_\nu, \tag{5.7}$$

where

$$e_\nu = \frac{c_0 b}{4a_0^2} e_{\nu_1}, \quad v = \frac{b}{a_0} v_1 \tag{5.8}$$

and k and λ are as defined previously. Note that, since $\alpha - 1 = O(\delta)$, $k = O(1)$.

† The statement $\sigma = O(\delta)$ should not be taken to imply any relation between the internal energy and the piston speed. A second parameter δ_1 such that $\sigma = O(\delta_1)$ can be introduced, and the subsequent analysis holds for terms $O(\delta, \delta_1)$, but all product terms, etc., are neglected.

Equations (5.6) and (5.7) can be combined to give

$$(v_\eta - vv_\xi)_\xi + \lambda(v_\eta - vv_\xi) + kv_\xi = 0. \tag{5.9}$$

It is apparent that both the high- and low-frequency factors (5.10) are governed by non-linear convective terms. In the near-frozen limit $\Lambda \to 0$ (i.e. $\lambda, k \to 0$) iterating on (5.9) shows that

$$v_\eta - vv_\xi + kv = 0 \tag{5.10}$$

to $O(\Lambda)$. Equation (5.10) agrees with (4.3), which was derived directly from the full equations. The near-equilibrium behaviour $\Lambda \to \infty$ $(\lambda, k \to \infty)$ can also be found by iteration. Including terms $O(\Lambda^{-1})$ (5.9) gives

$$v_\eta - vv_\xi + \frac{k}{\lambda} v_\xi = \frac{k}{\lambda^2} v_{\xi\xi}.$$

It is more appropriate to use the equilibrium variable $\bar{\xi}$, and this last result becomes Burger's equation

$$v_\eta - vv_{\bar{\xi}} = (k/\lambda^2)\, v_{\bar{\xi}\bar{\xi}}. \tag{5.11}$$

Since $$b = \bar{b} + O(\delta), \quad \frac{k}{\lambda^2} = \frac{\Gamma}{2}\frac{\alpha - 1}{\delta} + O(\delta)$$

(5.11) agrees with (3.4) when only the dominant terms, with respect to δ, are retained.

Equations (5.10) and (5.11) can also be deduced, for arbitrary Λ, by means of suitable co-ordinate expansions for ξ small and for ξ and η large respectively. This again agrees with the known behaviour of the full equation.

The linearized form of (5.9), namely

$$v_{\eta\xi} + \lambda v_\eta + kv_\xi = 0, \tag{5.12}$$

is the telegraph equation and has previously been used in discussing non-equilibrium flows by Moore & Gibson (1960). Moore & Gibson derived their equation from the linearized result (2.12) assuming that the difference in sound speeds was small. If their result is to have some validity it is apparent that not only must $(\alpha - 1) \ll 1$ but also

$$\delta^{-1}(\alpha - 1) \gg 1, \tag{5.13}$$

otherwise higher-order terms in the expansion outlined in § 2 may be important. It is worth noting that in Moore & Gibson's formulation $x = O((\alpha - 1)^{-1})$.

Although this linear equation may be of interest, it should be stressed that, within the framework of the approximations discussed here, it is not a valid approximation when ξ and η are $O(1)$. It does hold, however, if the supplementary restriction (5.13) applies, i.e. $\delta_1 \gg \delta$ (Spence & Ockendon 1968).

The parameters λ and k can be eliminated from (5.9) by putting

$$v = \frac{k}{\lambda} w, \quad \xi = \frac{\psi}{\lambda}, \quad \eta = \frac{Y}{k}, \tag{5.14}$$

and hence $$(w_Y - ww_\psi)_\psi + w_Y - ww_\psi + w_\psi = 0. \tag{5.15}$$

It is sometimes convenient to study the equation in this form. In these variables the initial condition (5.1) is

$$w = \frac{\lambda b}{ka_0} f'\left(\frac{\psi}{\lambda}\right) \quad \text{on} \quad Y = 0. \tag{5.16}$$

For geometrically similar bodies (5.15) and (5.16) imply that the solutions in the far field are similar for fixed values of the parameters λ and b/ka_0. Note that this result holds without making any detailed specifications about the enthalpy $h(p, \rho, \sigma)$ and the rate function $F(p, \rho, \sigma)$. Moreover, for power law piston paths the solution depends only on a single parameter.

For impulsive (constant speed) piston motions the solution, outside of any centred wave, is governed by the parameter $\lambda b/ka_0$. Within a centred expansion fan the condition at the origin replacing (5.16) is

$$w \sim -\psi/Y \tag{5.17}$$

and the solution is independent of all parameters. This solution is discussed further in § 7.

6. Steady-state solutions

Although general solutions of (5.15) are not readily found analytically for arbitrary piston motions, solutions of steady-state form

$$w = w(\psi + CY) = w(r) \tag{6.1}$$

can be deduced. These solutions represent the asymptotic state due to a piston moving at constant speed. The wave speed associated with (6.1) is, in (x, t) space,

$$U_w = a_0[1 + \delta(2kC/\lambda)] \approx a_0[1 + (\alpha - 1)C] \tag{6.2}$$

correct to $O(\delta)$.

Substitution into (5.9) gives either

$$\Omega = 0 \tag{6.3}$$

or

$$(d/dW)\{(1-W)\Omega\} = W - A, \tag{6.4}$$

where

$$W = C^{-1}w, \quad \Omega = C^{-1}w' \tag{6.5}$$

and

$$A = 1 + C^{-1}. \tag{6.6}$$

From (6.4) it follows that in general

$$\Omega = \frac{\frac{1}{2}W^2 - AW}{1-W} + \frac{K}{1-W} \tag{6.7}$$

and, since $\Omega = W = 0$ at upstream infinity, apparently

$$K = 0. \tag{6.8}$$

However, the corresponding solution from (6.7) is unique only if

$$A < \tfrac{1}{2}, \tag{6.9}$$

and the piston speed is given by

$$w_p = CW_p = 2(C+1). \tag{6.10}$$

This latter result also follows from conservation arguments, assuming that equilibrium is achieved on the piston face.

For compression waves, $w_p > 0$, (6.9) and (6.10) imply that

$$0 > C > -1. \tag{6.11}$$

Consequently, from (6.2),
$$a_0 > U_w > \bar{a}_0, \tag{6.12}$$

which is the usual condition for a fully dispersed wave (Lighthill 1956).

Equation (6.7), with $K = 0$, also appears to provide a valid solution for expansion waves, $w_p < 0$, when (6.9) holds. However, it is easily shown that the overall entropy change between the limiting upstream and downstream equilibrium states is

$$\frac{1}{3T_0}\frac{\bar{b}}{\bar{a}}w_p^3, \tag{6.13}$$

where T is the (non-dimensional) translational temperature (see Hayes 1958). Since the coefficient of w_p^3 is always positive it follows that only the fully dispersed compression wave solutions are admissible.

If $A > \frac{1}{2}$, equation (6.7) ($K = 0$) does not represent a single-valued solution. For compression waves it is necessary to insert a Rankine–Hugoniot shock at the wave front. Immediately behind the shock, across which $[\sigma] = 0$,

$$W = 2 \tag{6.14}$$

(see (4.13) and (6.2)). In addition, from (5.6) and (5.7) with $[\sigma] = 0$,

$$\Omega = 2/C \tag{6.15}$$

behind the shock front. The solution upstream of the shock is now defined by (6.3). Behind the shock the general solution is of the form (6.7) but (6.14) and (6.15) again imply that $K = 0$. As before, the piston speed is given by (6.10).

Note that (6.7), with $K = 0$, can be integrated to give

$$Ar = \log\left\{\left(\frac{A-1}{A-\frac{1}{2}W}\right)^{2A-1}\frac{2}{W}\right\}. \tag{6.16}$$

For expansion waves, with $A > \frac{1}{2}$, the asymptotic disturbance will be the equilibrium centred wave

$$w = -\frac{\psi}{Y}+1 = -\frac{\bar{\psi}}{Y}, \tag{6.17}$$

which precedes any steady-state region governed by (6.7). Equation (6.17) is an exact solution of (5.15). Consequently on the tail of the fan, from (6.17) and (6.1),

$$W = A. \tag{6.18}$$

Further, it can be shown (see e.g. Jones 1964) that Ω is exponentially small on the tail. Therefore from (6.7)

$$K = \tfrac{1}{2}A^2, \tag{6.19}$$

and hence from (6.7)
$$W = A, \quad \Omega = 0. \tag{6.20}$$

Thus, as expected, the appropriate solution downstream of the fan is a region of constant state.

7. Characteristic co-ordinates and shocks

(*a*) *Non-centred waves*

The characteristics of the non-linear equation (5.15) are defined by Y = constant and Φ = constant where

$$\left(\frac{\partial\psi}{\partial Y}\right)_\Phi = -w. \tag{7.1}$$

For a piston which starts from rest with a finite acceleration, Φ can be identified with ψ on $Y = 0$, or

$$\psi = \Phi - \int_0^Y w(\Phi, s)\, ds. \tag{7.2}$$

When Φ and Y are used as independent variables and ψ as the dependent variable, (5.15) becomes

$$\psi_{\Phi YY} + \psi_\Phi \psi_{YY} + \psi_{\Phi Y} = 0, \tag{7.3}$$

subject to the front condition

$$\psi(0, Y) = 0 \tag{7.4}$$

and the initial conditions

$$\psi(\Phi, 0) = \Phi, \quad \psi_Y(\Phi, 0) = \frac{-\lambda b}{ka_0} f'\left(\frac{\Phi}{\lambda}\right). \tag{7.5}$$

If the solution is regular in the characteristic plane and is known up to some line $\Phi = \Phi_0$, on which $\psi = \psi_0(Y)$, it can be extended across the line (i.e. normal derivatives can be found) by means of the formal expansion

$$\psi = \sum_{n=0} (\Phi - \Phi_0)^n \psi_n(Y), \tag{7.6}$$

where, in particular, ψ_1 satisfies

$$\psi_1'' + \psi_1' + \psi_0''\psi_1 = 0, \tag{7.7}$$

with

$$\psi_1(0) = 1, \quad \psi_1'(0) = -l_0', \tag{7.8}$$

where

$$l_0^{(n)} = (b/ka_0)\, \lambda^{-(n+1)} f^{(n+1)}(\Phi_0/\lambda). \tag{7.9}$$

It is also useful to note the relations

$$\psi_0'(0) = -l_0, \quad \psi_0''(0) = e^{-\Phi_0}\int_0^{\Phi_0} l_0(t)\, e^t\, dt. \tag{7.10}$$

The most direct application of this approach is at the front $\Phi_0 = 0$. It is easily shown that the expansion yields the usual results there (Rarity 1967) and, if $f'(\phi)$ is replaced by ϕ, is consistent with the behaviour discussed in § 4. In § 4 it was noted that the near-frozen expansion is not uniformly valid as $Y \to \infty$. A similar remark is true, in the usual asymptotic sense, for the co-ordinate expansion (7.6) near the front; convergence is governed by the magnitude of the product ΦY.

The solution of (7.3) is unique in the physical (ψ, Y) space only if the Jacobian of the transformation does not vanish, i.e.

$$\psi_\Phi \neq 0 \tag{7.11}$$

or, from (7.6) and (7.8),

$$\psi_1 > 0. \tag{7.12}$$

At the front this condition is equivalent to (4.12) and, when (7.12) does hold there, it is of interest to discuss constraints on the piston motion for which the remainder of the flow will be shock-free. As a fairly general example suppose that the piston accelerates smoothly to some constant speed. In this case both l_0 and l_0' are positive and bounded.

It is apparent from (7.2) that if $w_\Phi < 0$, (7.11) is certainly true. Since $w_\Phi > 0$ in the neighbourhood of the front ($l_0 > 0$) it is sufficient to consider only this possibility. From (5.6), (5.7) and (5.14) it can be shown that

$$w_Y = -e^{-\psi(\Phi, Y)} \int_0^\Phi e^{\psi(s, Y)} w_s \, ds \tag{7.13}$$

and hence, for $w_\Phi > 0$, $w_Y < 0$. Moreover, since $w > 0$ sufficiently near to the front it follows that $w > 0$ throughout the region of interest. Consequently, on $\Phi = \Phi_0$ within this domain, $\psi_0' < 0$ and $\psi_0'' > 0$.

Equation (7.6), together with the boundary conditions (7.8), can be replaced by the Volterra integral equation

$$\psi_1 = L(Y) - \int_0^Y K(s, Y)\, \psi_1(s)\, ds, \tag{7.14}$$

where
$$L(Y) = 1 - l_0'(1 - e^{-Y}) \tag{7.15}$$

and
$$K(s, Y) = \psi_0''(s)\,[1 - e^{-(Y-s)}]. \tag{7.16}$$

There is obviously a finite interval in Y for which $\psi_1 > 0$. Let $Y = Y^*$ lie within this interval. Since the kernel is non-negative (and $l_0 > 0$)

$$1 > \psi_1(Y^*) > L(Y^*) - \int_0^{Y^*} \psi_0''(s)\, ds = L(Y^*) - \psi_0'(Y^*) - l_0$$

and ψ_1 will certainly remain positive for all Y if

$$1 > l_0 + l_0'. \tag{7.17}$$

This simple result agrees with (4.12) on the front, but it should be stressed that it is a sufficient rather than a necessary condition. The necessary and sufficient condition (7.12) can be replaced by a formal solution of the integral equation (7.14) though the result does not appear to be particularly useful unless ψ_0 is known analytically.

It is also fairly straightforward to argue conversely and show that shocks will certainly form if

$$l_0' > 1, \tag{7.18}$$

which is identical with the result of § 4 for the front.

(*b*) *Centred waves*

In this case (7.2) is replaced by

$$\psi = -\int_0^Y w(\Phi, s)\, ds \tag{7.19}$$

and, for convenience, Φ is defined by the slope at the origin $\psi = Y = 0$. Hence

$$w(\Phi, 0) = -\Phi. \tag{7.20}$$

Within the fan (7.7) again holds, though the initial conditions (7.8) are replaced by

$$\psi_1(0) = 0, \quad \psi_1'(0) = 1, \tag{7.21}$$

and it is useful to note that

$$\psi_0'(0) = \Phi_0, \quad \psi_0''(0) = -\Phi_0. \tag{7.22}$$

Some recent calculations (see Mohammad 1967) have suggested the existence of shocks at the tail of non-equilibrium centred expansion waves. Although their occurrence was confined essentially to large-amplitude waves it is of interest to discuss the possibility of shock formation both within and downstream of the fan in the present limit. Shocks will again occur where, apart from the origin, $\psi_\Phi = 0$.

Within the fan it is easily shown, both in the neighbourhood of the front and near the corner, that $w_\Phi < 0$ and hence, from (7.19), that $\psi_\Phi > 0$ ($Y > 0$). In fact, if it can be established that w_Φ remains negative, the solution in the interior of the fan is unique. Moreover, in such a domain $w_Y > 0$ (see (7.13)).

Suppose $\Phi = \Phi^*$ is the first value of Φ at which $w_\Phi = 0$, and $Y = Y^*$ is the corresponding value of Y, then, on $\Phi = \Phi^*$, $\psi_0'' < 0$ ($Y \leqslant Y^*$), $\psi_1'(Y^*) = 0$ and (7.6) implies that

$$\operatorname{sgn}\{\psi_1''(Y^*)\} = \operatorname{sgn}\{\psi_1(Y^*)\}. \tag{7.23}$$

However, the boundary conditions (7.21) imply that Y^* corresponds to a maximum at which $\psi_1 > 0$. This result is not consistent with (7.23) and hence the original supposition is incorrect. Thus $w_\Phi < 0$ and the solution is unique. One consequence of this result is that the energy does not overshoot its equilibrium value.

Downstream of the fan $\Phi \geqslant \Phi_t$, where the subscript t denotes the tail of the fan, $\psi(\Phi, Y)$ is defined by

$$\psi = \Phi - \Phi_t - \int_0^Y w(\Phi, s)\, ds. \tag{7.24}$$

If the expansion (7.6) is applied in this region, ψ_1 obviously satisfies (7.7) but the initial conditions become (constant piston speed)

$$\psi_1(0) = 1, \quad \psi_1'(0) = 0. \tag{7.25}$$

The solution within the fan implies that $\psi_t'' < 0$. Therefore, from (7.7) and (7.25), $\psi_1''(0) > 0$ and $\psi_1 = 0$ can be attained only if ψ_1 passes through a maximum. It is easily shown that this leads to a contradiction.

It follows that in the neighbourhood of the tail there is a region in which $w_\Phi < 0$. A similar inequality can also be established sufficiently close to the piston face. A repetition of the argument used within the fan shows that the solution is single-valued downstream of the fan in the (ψ, Y) plane. (From (7.24), if $w_\Phi < 0$, $\psi_\Phi > 0$.)

(*c*) *Numerical example*

Conditions under which the solution was regular, or otherwise, in physical space were established in § 7 (*a*) and (*b*) under the assumption that the solution was regular in the characteristics plane. To compute the solution in the (Φ, Y) plane it is convenient to replace (7.3) by the original three first-order relations

$$\left(\frac{\partial w}{\partial Y}\right)_\Phi = E - w, \tag{7.26}$$

$$\left(\frac{\partial \psi}{\partial Y}\right)_\Phi = -w, \tag{7.27}$$

$$\left(\frac{\partial E}{\partial \Phi}\right)_Y = \left(\frac{\partial \psi}{\partial \Phi}\right)_Y (w - E), \tag{7.28}$$

where

$$e_\nu = (k^2/\lambda^2)\, E. \tag{7.29}$$

It is apparent, subject to the boundary conditions on w, E and ψ outlined earlier, that these relations can be used to find appropriate derivatives and the solution extended into the domain $\Phi > 0$, $Y > 0$.

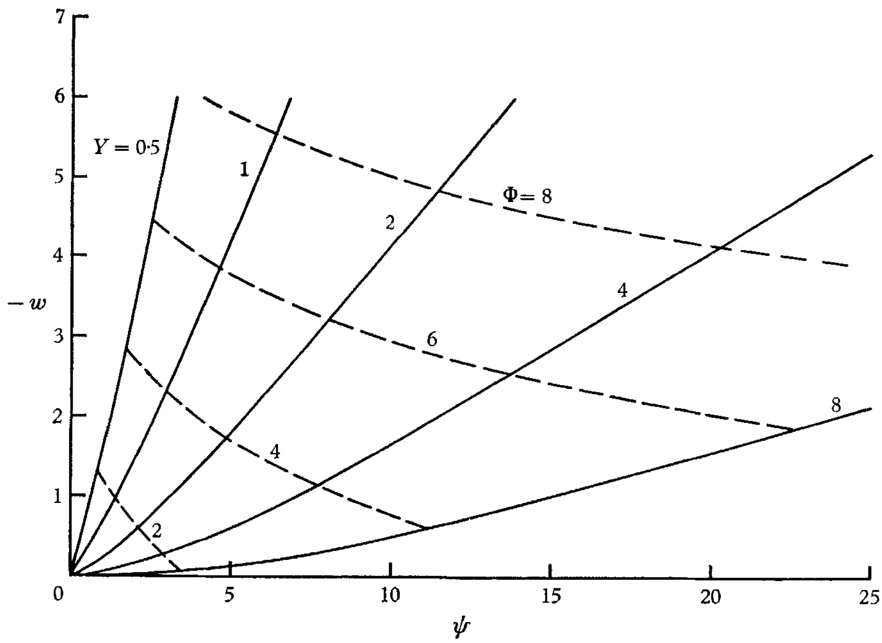

FIGURE 2. Solution within the fan for a centred wave.

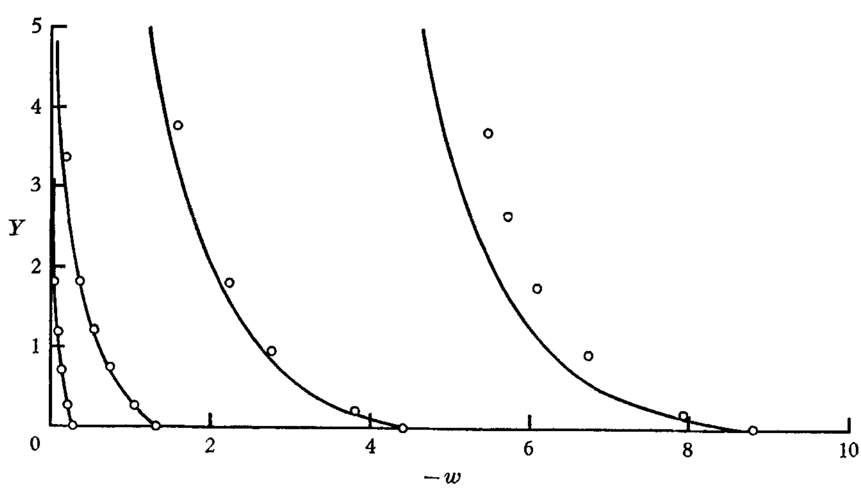

FIGURE 3. Comparison with exact numerical calculations (Johannesen 1968) for the velocity variation along a characteristic line. —, approximate theory; ○, Johannesen.

A suitable finite-difference scheme has been used to obtain the solution for a centred expansion wave. The solution in the (ψ, Y) plane is easily found from the characteristics plane solution (it was proved in § 7 (*b*) that the former solution is single-valued), and the results are displayed in figure 2.

It was noted in § 5 that in terms of these variables the solution is independent of the parameters. Unfortunately, available calculations were not sufficiently detailed to test this similarity rule but a comparison with an exact calculation for a vibrationally relaxing gas is shown in figure 3 (Johannesen 1968).

8. Steady two-dimensional flow

If the variables are suitably re-interpreted many of the previously derived results apply equally well to steady two-dimensional supersonic† flow past thin bodies. The appropriate formulation in the small energy far-field limit is given below.

In deriving the governing equations it is simplest to use streamline co-ordinates and to express the equations in the corresponding characteristic form (see § 2). By stretching the (non-dimensional) dependent and independent variables in a way similar to that outlined in § 5, it is easy to show that the usual relationships

$$p_1 = a_0^2 \rho_1 = -U_0 u_1 = U_0(M_0^2-1)^{-\frac{1}{2}} v_1 \tag{8.1}$$

hold, where (u_1, v_1) are the perturbation velocity components in a Cartesian co-ordinate system (x, y). The chord line of the body is identified with $y = 0$. U_0 is the free-stream speed, which is directed along the x-axis, and M_0 is the free-stream Mach number. It can also be shown that the velocity and energy perturbations, v and e_ν respectively, again satisfy (5.6) and (5.7) with λ and k replaced by λ^* and k^*, where

$$\lambda^* = \lambda / U_0, \quad k^* = \frac{\alpha^2 - 1}{4\delta\alpha^2}\lambda^*. \tag{8.2}$$

However, in this section

$$\xi = x - (M_0^2-1)^{-\frac{1}{2}} y, \quad \eta = 2(M_0^2-1)^{\frac{1}{2}}\delta y \tag{8.3}$$

and

$$v = \frac{b}{a_0}\frac{M_0^3}{(M_0^2-1)^{\frac{1}{2}}} v_1, \quad e_\nu = \frac{1}{4}\frac{bc_0}{a_0^2} M_0^4 e_{\nu_1}. \tag{8.4}$$

The body shape is defined by $y = \delta f(x)$ (8.5)

and the appropriate matching condition is

$$v_1 = U_0 f'(\xi) \quad \text{on} \quad \eta = 0. \tag{8.6}$$

These transformations establish the detailed correspondence between the unsteady and steady flows considered here. In addition, by putting

$$v = \frac{k^*}{\lambda^*} w, \quad \xi = \frac{\psi}{\lambda^*}, \quad \eta = \frac{Y}{k^*}, \tag{8.7}$$

the equations can again be combined to yield (5.15) with

$$w = \frac{b\lambda^*}{k^*}\frac{M_0^4}{\sqrt{(M_0^2-1)}} f'\left(\frac{\psi}{\lambda^*}\right). \tag{8.8}$$

The similarity laws discussed in § 5 also apply here and (8.8) specifies the appropriate similarity parameters. In particular, the solution obtained in § 7 (*c*) also governs the non-equilibrium supersonic flow round a sharp corner, where w, ψ and Y are now to be interpreted according to (8.3), (8.4) and (8.7).

† With respect to the frozen sound speed.

The author would like to thank Prof. N. H. Johannesen for making available to him the characteristic calculations for the full equations, which were discussed in § 7. He is also indebted to Mr D. G. Petty for his assistance in programming the numerical calculation presented in that section.

Appendix

If d' is some representative length, the independent variables are non-dimensionalized by writing

$$x = \frac{x'}{d'}, \quad t = \left(\sqrt{\frac{p_0'}{\rho_0'}}\right)\frac{t'}{d'},$$

where primes denote dimensional quantities.

The dependent variables are normalized by putting

$$p = p'/p_0', \quad \rho = \rho'/\rho_0', \quad u = u' \Big/ \sqrt{\left(\frac{p_0'}{\rho_0'}\right)},$$

and

$$h = \frac{\rho_0' h'}{p_0'}, \quad \sigma = \frac{\rho_0' \sigma'}{p_0'}.$$

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